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Teaching Mathematics for Understanding

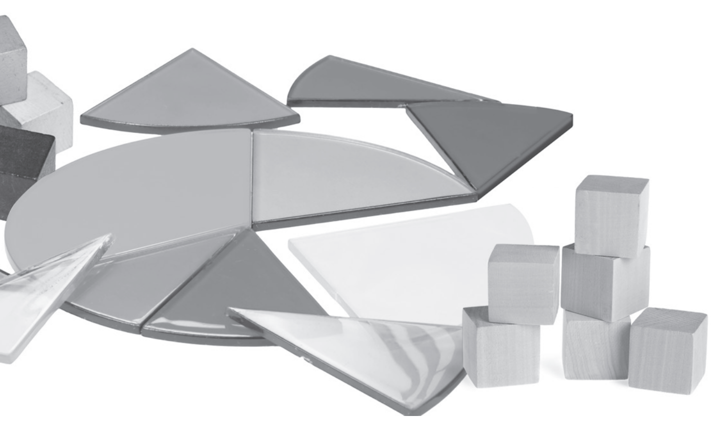
An understanding can never be "covered" if it is to be understood.

Wiggins and McTighe (2005, p. 229)

Teachers generally agree that teaching for understanding is a good thing. But this statement begs the question: What is understanding? Understanding is being able to think and act flexibly with a topic or concept. It goes beyond knowing; it is more than a collection of information, facts, or data. It is more than being able to follow steps in a procedure. One hallmark of mathematical understanding is a student's ability to justify why a given mathematical claim or answer is true or why a mathematical rule makes sense (Council of Chief State School Officers [CCSSO], 2010). Although students might know their basic multiplication facts and be able to give you quick answers to questions about these given facts, they might not understand multiplication. They might not be able to justify how they know an answer is correct or provide an example of when it would make sense to use this basic fact. These tasks go beyond simply knowing mathematical facts and procedures. Understanding must be a primary goal for all of the mathematics you teach.

Understanding and Doing Mathematics

Procedural proficiency, a main focus of mathematics instruction in the past, remains important today, but conceptual understanding is an equally important goal (National Council of Teachers of Mathematics [NCTM], 2000; National Research Council, 2001; CCSSO, 2010). Numerous reports and standards emphasize the need to address skills and understanding in an integrated manner; among these are the *Common Core State Standards* (CCSSO, 2010), a state-led effort coordinated by the National Governors Association Center for Best Practices (NGA Center)



and CCSSO, which has been adopted by nearly every state and the District of Columbia. This effort has resulted in attention to *how* mathematics is taught, not just *what* is taught.

The National Council of Teachers of Mathematics (NCTM, 2000) identifies the process standards of problem solving, reasoning and proof, representation, communication, and connections as ways to think about how students should engage in learning the content as they develop both procedural fluency and conceptual understanding. Students engaged in the process of *problem solving* build mathematical knowledge and understanding by grappling with and solving genuine problems, as opposed to completing routine exercises. They use *reasoning and proof* to make sense of mathematical tasks and concepts and to develop, justify, and evaluate mathematical arguments and solutions. Students create and use *representations* (e.g., diagrams, graphs, symbols, and manipulatives) to reason through problems. They also engage in *communication* as they explain their ideas and reasoning verbally, in writing, and through representations. Students develop and use *connections* between mathematical ideas as they learn new mathematical concepts and procedures. They also build *connections* between mathematics and other disciplines through applying mathematics to real-world situations. By engaging in these processes, students are learning mathematics by *doing* mathematics. Consequently, the process standards should not be taught separately from but in conjunction with mathematics as ways of learning mathematics.

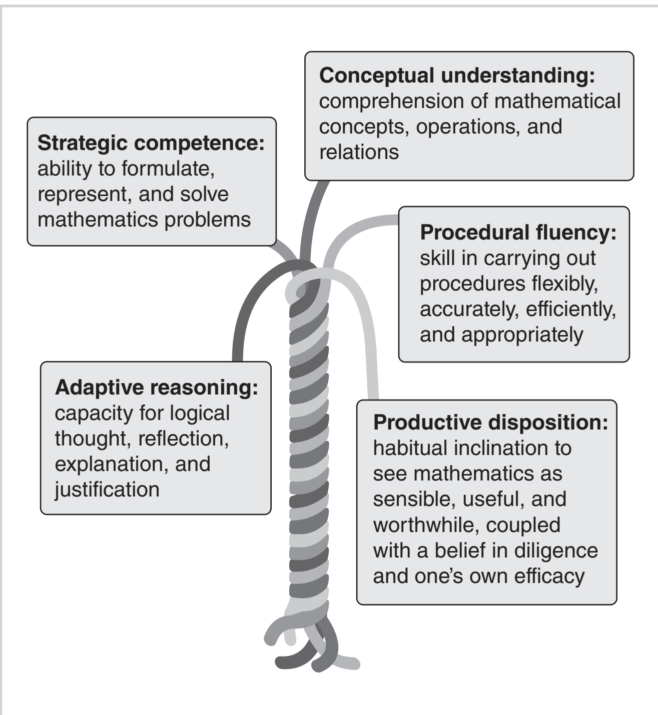
Adding It Up (National Research Council, 2001), an influential research review on how students learn mathematics, identifies the following five strands of mathematical proficiency as indicators that someone understands (and can do) mathematics.

- *Conceptual understanding:* Comprehension of mathematical concepts, operations, and relations
- *Procedural fluency:* Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *Strategic competence:* Ability to formulate, represent, and solve mathematical problems
- *Adaptive reasoning:* Capacity for logical thought, reflection, explanation, and justification
- *Productive disposition:* Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Reprinted with permission from p. 116 of *Adding It Up: Helping Children Learn Mathematics*, 2001 by the National Academy of Sciences, Courtesy of the National Academies Press, Washington, D.C.)

This report maintains that the strands of mathematical proficiency are interwoven and interdependent—the development of one strand aids the development of others (Figure 1.1).

Figure 1.1

Interrelated and intertwined strands of mathematical proficiency.



Source: Reprinted with permission from Kilpatrick, J., Swafford, J., & Findell, B. (Eds.), *Adding It Up: Helping Children Learn Mathematics.* Copyright 2001 by the National Academy of Sciences. Courtesy of the National Academies Press, Washington, D.C.

Building on the NCTM process standards and the five strands of mathematical proficiency, the *Common Core State Standards* (CCSSO, 2010) outline the following eight Standards for Mathematical Practice (see Appendix A) as ways in which students can develop and demonstrate a deep understanding of and capacity to do mathematics. Keep in mind that you, the teacher, have a responsibility in helping students develop these practices. Here we provide a brief discussion about each mathematical practice.

1. *Make sense of problems and persevere in solving them.* To make sense of problems, students need to learn how to analyze the given information, parameters, and relationships in a problem so that they can understand the situation and identify possible ways to solve it. One way to help students analyze problems is to have them create proportional drawings to make sense of the quantities and relationships involved. Once students learn various strategies for making sense of problems, encourage them to remain committed to solving them. As they learn to monitor and assess their progress and change course as needed, they will solve the problems they set out to solve!

2. *Reason abstractly and quantitatively.* This practice involves students reasoning with quantities and their relationships in problem situations. You can support students' development of this practice by helping them create representations that correspond to the meanings of the quantities and the units involved. Also, a significant aspect of this practice is to be able to represent and manipulate the situation symbolically. Encourage students to find connections between the abstract symbols and the representation that illustrates the quantities and their relationships. For example, suppose students are generalizing a situation in which they are trying to determine what their earnings at the local ice-cream store will be if they earn $8.00 for each hour they work but spend $2.50 every two hours for their own ice-cream cones. They may represent the relationship symbolically as $y = 8x - 2.5(\frac{x}{2})$ or as $y = (8 - 1.25)x$ or as $y = 6.75x$. Ultimately, students should be able to reason how these equations are equal and relate the equations to the situation.

3. *Construct viable arguments and critique the reasoning of others.* This practice emphasizes the importance of students using mathematical reasoning as the basis for justifying their ideas and solutions, including being able to recognize and use counterexamples. Encourage students to examine each others' arguments to determine whether they make sense and to identify ways to clarify or improve the arguments. This practice emphasizes that mathematics is based on reasoning and should be examined in a community—not carried out in isolation. Tips for supporting students as they learn to justify their ideas can be found in Chapter 2.

4. *Model with mathematics.* This practice encourages students to use the mathematics they know to solve problems in everyday life, and to be able to represent them symbolically (i.e., the equation serves as a model of the situation). The equations given above for the ice-cream shop are models for describing the students' earnings. The equation (model) can then be used to predict and find earnings for any number of hours worked. Be sure to encourage students to determine whether the mathematical model is the generalization of the situation.

5. *Use appropriate tools strategically.* Students should become familiar with a variety of visuals and tools that can be used to solve a problem and they should learn to choose which ones are most appropriate for a given situation. For example, suppose students have used the following tools to investigate probability: coins, spinners, number cubes, and computerized simulations. If students are asked to create a simulation with two outcomes, one outcome twice as likely as the second outcome, they should consider which of these tools can best support their simulation. If the number cubes are six-sided cubes, the students might define one outcome as rolling a 1, 2, 3, or 4 and the second outcome as rolling a 5 or 6. However, a coin, because it is two-sided, would not be an appropriate tool for this particular investigation because the outcomes could not be modified to model the given situation.

Teaching Tip

Research suggests that students, in particular girls, may tend to continue to use the same tools because they feel comfortable with the tools and are afraid to take risks (Ambrose, 2002). Look for students who tend to use the same tool or strategy every time they work on tasks. Encourage all students to take risks and try new tools and strategies.

6. *Attend to precision.* In communicating ideas to others, it is imperative that students learn to be explicit about their reasoning. For example, they need to be clear about the meanings of the operations and symbols they use, to indicate the units involved in a problem, and to clearly label the diagrams they provide in their explanations. As students share their ideas, make this expectation clear and ask clarifying questions that help make the details of their reasoning more apparent. Teachers can further encourage students' attention to precision by introducing, highlighting, and encouraging the use of accurate mathematical terminology in explanations and diagrams.

7. *Look for and make use of structure.* Students who look for and recognize a pattern or structure can experience a shift in their perspective or understanding. Therefore, set the expectation that students will look for patterns and structure, and help them reflect on their significance. For example, when students begin to write rational numbers in decimal form, they learn that the decimal either terminates or repeats. Look for opportunities to help students notice that the denominator of any simplified rational number whose decimal form terminates can be rewritten as a power of ten—which provides insight into why the decimal terminates.

8. *Look for and express regularity in repeated reasoning.* Encourage students to step back and reflect on any regularity that occurs in an effort to help them develop a general idea or method to identify shortcuts. For example, as students begin adding signed numbers, they will encounter situations such as $5 - 3$ and $5 + (-3)$. Over time, help them reflect on the results of these situations. Eventually they should be able to express that subtracting a positive number is equivalent to adding the opposite (or negative) of the number.

Like the process standards, the Standards for Mathematical Practice should not be taught separately from the mathematics, but should instead be incorporated as ways for students to learn and do mathematics. Students who learn to use these eight mathematical practices as they engage with mathematical concepts and skills have a greater chance of developing conceptual understanding. Note that learning these mathematical practices, and consequently developing understanding, takes time. So the common notion of simply and quickly "covering the material" is problematic. The opening quotation states it well: "An understanding can never be 'covered' if it is to be understood" (Wiggins & McTighe, 2005, p. 229). Understanding is an end goal—that is, it is developed over time by incorporating the process standards and mathematical practices and striving toward mathematical proficiency.

How Do Students Learn?

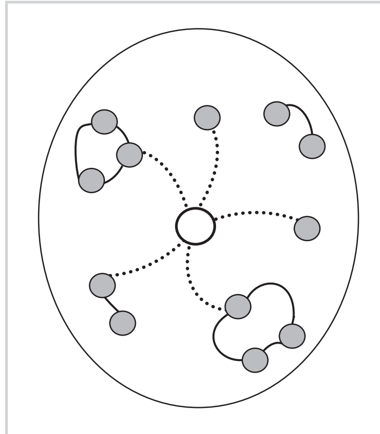
Let's look at a couple of research-based theories that can illustrate how students learn in general: constructivism and sociocultural theory. Although one theory focuses on the individual learner while the other emphasizes the social and cultural aspects of the classroom, these theories are not competing; they are actually compatible (Norton & D'Ambrosio, 2008).

Constructivism

At the heart of constructivism is the notion that learners are not blank slates but rather creators (constructors) of their own learning. All people, all of the time, construct or give meaning to things they perceive or think about. Whether you are listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge (existing schemas) to make sense of the new information.

Constructing something in the physical world requires tools, materials, and effort. The tools you use to build understanding are your existing ideas and knowledge. Your materials might be things you see, hear, or touch, or they might be your own thoughts and ideas. The effort required to construct knowledge and understanding is reflective thought.

Figure 1.2
How someone constructs a new idea.



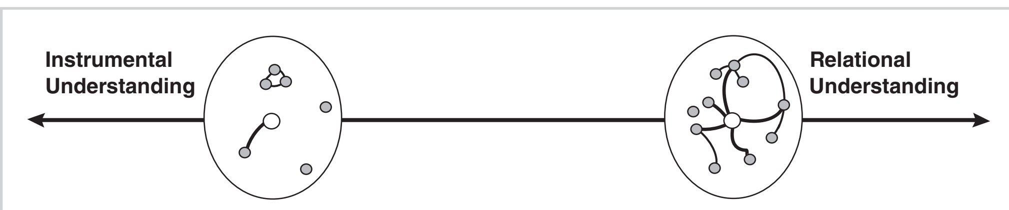
Through reflective thought, people connect existing ideas to new information and thus modify their existing schemas or background knowledge to incorporate new ideas. Making these connections can happen in either of two ways—*assimilation* or *accommodation*. Assimilation occurs when a new concept "fits" with prior knowledge and the new information expands an existing mental network. Accommodation takes place when the new concept does not "fit" with the existing network, thus creating a cognitive conflict or state of confusion that causes what theorists call *disequilibrium*. As an example, consider what happens when students start learning about variables. They begin in elementary school by using variables as unknowns, as in $2 + ? = 5$ or $4 \times ? = 24$, in which their goal is to determine what the question mark represents. Consequently, some students come to see a variable as a placeholder or missing number. So, when students encounter equations such as $y = 4x + 5$, the variable does not represent a single missing number (assimilation), but rather many values. Students must adapt their mental image of what a variable means (accommodation). It is through the struggle to resolve the disequilibrium that the brain modifies or replaces the existing schema so that the new concept fits and makes sense, resulting in a revision of thought and a deepening of the learner's understanding.

For an illustration of what it means to construct an idea, consider Figure 1.2. The gray and white dots represent ideas, and the lines joining the ideas represent the logical connections or relationships that develop between ideas. The white dot is an emerging idea, one that is being constructed. Whatever existing ideas (gray dots) are used in the construction are connected to the new idea (white dot) because those are the ideas that give meaning to the new idea. The more existing ideas that are used to give meaning to the new one, the more connections will be made.

Each student's unique collection of ideas is connected in different ways. Some ideas are well understood and well formed (i.e., connected), and others are less so as they emerge and students build connections. Students' experiences help them develop connections and ideas about whatever they are learning.

Understanding exists along a continuum (Figure 1.3) from an instrumental understanding—knowing something by rote or without meaning (Skemp, 1978)—to a relational understanding—knowing what to do and why. Instrumental understanding, at the left end of the continuum, shows that ideas (e.g., concepts and procedures) are learned, but in isolation (or nearly so) to other ideas. Here you find ideas that have been memorized. Due to their isolation, poorly understood ideas are easily forgotten and are unlikely to be useful for constructing new ideas. At the right end of the continuum is relational understanding. Relational understanding means that each new concept or procedure (white dot) is not only learned, but is also connected to many existing ideas (gray dots), so there is a rich set of connections.

Figure 1.3 Continuum of understanding.



A primary goal of teaching for understanding is to help students develop a relational understanding of mathematical ideas. Because relational understanding develops over time and becomes more complex as a person makes more connections between ideas, teaching for this kind of understanding takes time and must be a goal of daily instruction.

Sociocultural Theory

Like constructivism, sociocultural theory not only positions the learner as actively engaged in seeking meaning during the learning process, but it also suggests that the learner can be assisted by working with others who are "more knowledgeable." Sociocultural theory proposes that learners have their own zone of proximal development, which is a range of knowledge that may be out of reach for the individuals to learn on their own but is accessible if learners have the support of peers or more knowledgeable others (Vygotsky, 1978). For example, when students are learning about experimental probability, they do not necessarily recognize the significance of sample size. Teachers often have students collect data to explore the probability of two events, such flipping two coins. Students may think that HH, TT, and HT (H=Heads; T=Tails) are equally likely and therefore each has a probability of $\frac{1}{3}$. A more knowledgeable person (a peer or teacher) will know that if the students explore a large number of trials, the data will suggest that HT actually has a 50 percent probability, and that creating a list of possible outcomes will help to understand why the probabilities are $\frac{1}{4}$, $\frac{1}{4}$, *and* $\frac{1}{2}$, respectively. The more knowledgeable person can draw students' attention to this critical idea of how possible outcomes connect to probability.

The best learning for any given student will occur when the conversation of the classroom is within his or her zone of proximal development. Targeting that zone helps teachers provide students with the right amount of challenge while avoiding boredom on the one hand and anxiety on the other when the challenge is beyond the student's current capability. Consequently, classroom discussions based on students' own ideas and solutions to problems are absolutely "foundational to children's learning" (Wood & Turner-Vorbeck, 2001, p. 186).

Teaching for Understanding

Teaching toward Relational Understanding

To explore the notion of understanding further, let's look into a learner-centered sixth-grade classroom. In learner-centered classrooms, teachers begin *where the students are*—with *the students*' ideas. Students are allowed to solve problems or to approach tasks in ways that make sense to them. They develop their understanding of mathematics because they are at the center of explaining, providing evidence or justification, finding or creating examples, generalizing, analyzing, making predictions, applying concepts, representing ideas in different ways, and articulating connections or relationships between the given topic and other ideas.

For example, in this sixth-grade classroom, the students are going to explore division of fractions by fractions. They have recently investigated multiplication of fractions and division of fractions by whole numbers and have used real contexts, manipulatives, and diagrams to make sense of the operations. Their work with multiplication of fractions has emphasized identifying the whole that is being used, an important idea in working with fractions and fraction computation. They also have revisited the different meanings of division: sharing or partitive division, and repeated subtraction or measurement division. The students have

had previous experiences dividing fractions by whole numbers and have not been taught the standard algorithm for division of fractions.

The teacher sets the following instructional objectives for the students:

1. Solve word problems involving the division of fractions by fractions by using diagrams.
2. Interpret and compute the quotients of fractions.

The lesson begins with a task that is designed to set the stage for the main part of the lesson. As is often the case, this class begins with a story problem to provide context and relevance to the mathematics. The teacher displays this problem on the board:

How much chocolate will each person get if 4 people share $\frac{1}{2}$ pound of chocolate equally?

Stop and Reflect

Which operation is being described in this situation? Before reading further, try to solve this problem without using the standard algorithm. As a hint, can you act it out? ■

Before students start working on the task, the teacher asks them to think about what operation is being used in the situation. After some wait time, some of the students explain that because the chocolate is being shared or divided equally among four people, fair sharing or partitive division is being used. Students are then given a few minutes to work on the problem with a partner, share their ideas with another group, and prepare to share their ideas and answers with the class. The following two ideas were most prominent among the strategies used:

- We cut the half of a pound into four equal pieces so that each person would have an equal share. But then we needed to figure out what those pieces were, so we extended our drawing to make the whole pound. Since there are eight equal pieces in the whole pound, one piece would be one-eighth. So each person would get one-eighth pound of chocolate (Figure 1.4).
- We pretended we had one pound of chocolate. Then each person would get one-fourth of a pound. But each person really gets only half of that since we started with one-half pound. We knew that one-fourth is the same as two-eighths. So one-eighth is half of one-fourth or two-eighths. So each person gets one-eighth pound of chocolate.

As students share their ideas, the teacher highlights the notion of sharing that is going on in each of the solutions as well as the attention given to the whole or the pound of chocolate.

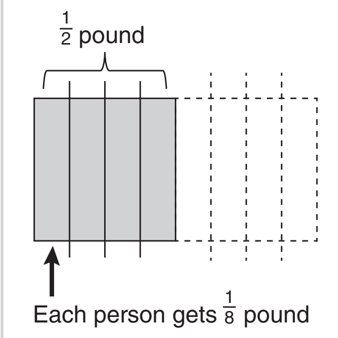
Because the teacher wants to extend students' thinking to division of fractions by fractions, she poses the following problem, which involves a fractional amount of money.

Dan paid $2\frac{1}{4}$ dollars for a $\frac{3}{5}$-pound box of candy. How much money is that per pound?

She uses money because she has found that students can easily think about partitioning money in a variety of ways. As is the norm in the class, students are told that they should be prepared to explain their reasoning with words and numbers as well as a drawing to support their explanation.

Figure 1.4

One solution for $\frac{1}{2} \div 4$.



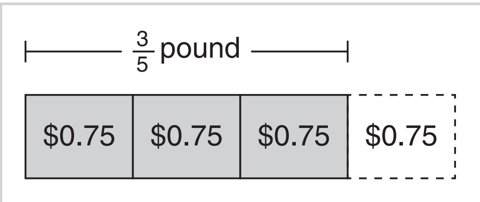
Stop and Reflect

Before reading further, how many different ways can you think of to solve the problem $2\frac{1}{4} \div \frac{3}{5}$? ■

The students work in pairs for about 15 minutes. The teacher listens to different students talk about the task and offers a hint to a few who are stuck. For example, she asks, "Did Dan buy an entire pound of candy? What ways can you partition the whole (the box of candy that Dan bought) to help you think about this problem?" Soon, the teacher begins a discussion by having students share their ideas and answers. As the students report, the teacher records their ideas on the board. Sometimes, the teacher asks questions to help clarify ideas for others. She makes no evaluative comments, although she asks the students who are listening if they understand or have any questions to ask the presenters. The following solution strategies are common in classes where students are regularly asked to generate their own approaches. Figure 1.5 shows a sketch for the first method shared below.

Figure 1.5

One solution for $2\frac{1}{4}$ dollars $\div$ $\frac{3}{5}$.



Group 1: Since we had $\frac{3}{5}$ pound, we drew three rectangles. We know that in $2\frac{1}{4}$ dollars we had nine fourths, so we put three-fourths of a dollar in each rectangle. Then, the whole pound would be two more three-fourths, or a total of $3\frac{3}{4}$ dollars or $3.75.

Teacher: What do the three rectangles represent in the problem?

Group 1: Each of the rectangles is $\frac{1}{5}$ pound. Since we had $\frac{3}{5}$ pound, we had to show three rectangles.

Teacher: And how did you get $3\frac{3}{4}$ dollars?

Group 1: We knew each rectangle or $\frac{1}{5}$ pound was seventy-five cents or three-fourths of a dollar. Since there are five $\frac{1}{5}$ pounds in one pound, we multiplied $\frac{3}{4}$ by 5.

Group 2: We used a circle and divided it into five parts but used only three of them since we only had $\frac{3}{5}$ pound of candy. We knew we could not put $1 in each part because we did not have $3, so we started with $0.50 (one-half dollar). That left $0.75 (or three-fourths of a dollar) to share, and we knew 75 divided by 3 is 25. So that meant there was $0.50 plus $0.25 or $0.75 in each section. To figure out how much the whole pound would be, we just added $0.75 to $2.25 to get $3.00 and then added $0.75 to $3.00 to get $3.75 or $3\frac{3}{4}$ dollars.

Stop and Reflect

The invert-and-multiply algorithm plays out in this kind of division problem (partitive or fair sharing). Can you find in the explanations above where the calculation $2\frac{1}{4} \times \frac{5}{3}$ occurs? (Hint: Think of $\frac{5}{3}$ as $\frac{1}{3} \times 5$.) ■

This vignette illustrates that when students are encouraged to solve a problem in their own way (using their own particular set of gray dots or ideas), they are able to make sense of their solution strategies and explain their reasoning. This is evidence of their development of mathematical proficiency.

During the discussion periods in classes such as this one, ideas continue to grow. The students may hear and immediately understand a clever strategy that they could have used but that did not occur to them. Others may begin to create new ideas to use that build from

thinking about their classmates' strategies over multiple discussions. Some in the class may hear excellent ideas from their peers that do not make sense to them. These students are simply not ready or do not have the prerequisite concepts (gray dots) to understand these new ideas. On subsequent days there will be similar opportunities for all students to grow at their own pace based on their own understandings.

Teaching toward Instrumental Understanding

In contrast to the lesson just described, in which students are developing concepts (understanding of fraction division) and procedures (ability to flexibly divide) and seeing the relationships between these ideas, let's consider how a lesson with the same basic objective (fraction division) might look if the focus is on instrumental understanding.

In this classroom, the teacher distributes coins to all students. The teacher reads to the class the same problem that was used in the first classroom about a $\frac{3}{5}$-pound box of candy costing \$2.25 or $2\frac{1}{4}$ dollars. She explains that this means that they will be dividing $2\frac{1}{4}$ by $\frac{3}{5}$ and that this is the same as flipping the second fraction and multiplying so that the problem becomes $2\frac{1}{4} \times \frac{5}{3}$. The teacher directs the students to count out $2\frac{1}{4}$ dollars from the coins (nine quarters). The discussion continues:

T: What is $2\frac{1}{4}$ divided by 3? Use your coins to help you find this amount.

S: [Students take some time to partition their coins] \$0.75.

T: So $2\frac{1}{4}$ dollars divided by 3 is \$0.75 or $\frac{3}{4}$ of a dollar. This means that each $\frac{1}{5}$ pound is worth \$0.75 or $\frac{3}{4}$ of a dollar. Can you see with our manipulatives that $\frac{3}{5}$ pound of candy costs $2\frac{1}{4}$ dollars or \$2.25 because we have three groups of \$0.75?

T: [pause] Now we need to find out how much one pound costs, and that's where we multiply the \$0.75 or $\frac{3}{4}$ of a dollar by 5. *One*-fifth of a pound is \$0.75, but there are *five* fifths in a whole pound, so we need to multiply \$0.75 by 5 to get \$3.75.

Next, the students are given three other similar problems to solve with manipulatives. They work in pairs and record their answers on their papers. The teacher circulates and helps anyone having difficulty by guiding them to use manipulatives and connect them to the steps in the standard algorithm.

In this lesson, the teacher and students are using manipulatives to illustrate the invert-and-multiply algorithm for fraction division. After engaging in several lessons similar to this one, most students are likely to remember, and possibly understand, how to divide fractions with the standard algorithm. Using manipulatives to illustrate the invert-and-multiply algorithm can build toward relational understanding. However, when the expectation is for all students to use one method, students do not have opportunities to apply other strategies that may help them build connections between subtraction and division, multiplication and division, or sharing (partitive) and repeated subtraction (measurement) concepts of division; these connections are fundamental characteristics of relational understanding. It is important to note that this lesson on the standard algorithm, in combination with other lessons that reinforce other approaches, *can* build a relational understanding, as it adds to students' repertoire of strategies. But if this lesson represents the sole approach to fraction division, then students are more likely to develop an instrumental understanding of mathematics.

The Importance of Students' Ideas

Let's take a minute to compare these two classrooms. By examining them more closely, you can see several important differences. These differences affect what is learned and who learns. Let's consider the first difference: Who determines the procedure to use?

In the first classroom, the students think about the meaning of division in the situation and the relationships between the numbers involved. Using this information, they generate a drawing of the situation to help them make sense of and solve the problem. So they *choose* a strategy that is based on *their* ideas, using what they know about subtraction, multiplication, and division. The students in the first classroom are being taught mathematics for understanding—*relational* understanding—and are developing the kinds of mathematical proficiency described earlier.

In the second classroom, the teacher provides one strategy for how to divide fractions—the standard algorithm. Although the standard algorithm is a valid strategy, the entire focus of the lesson is on the steps and procedures that the teacher has outlined. The teacher solicits no ideas from individual students about how to partition the numbers and instead is only able to find out who has or has not been able to follow directions.

When students have more choice in determining which strategies to use, as in the first classroom, they can learn more content and make more connections. In addition, if teachers do not seek out and value students' ideas, students may come to believe that mathematics is a body of rules and procedures that are learned by waiting for the teacher to tell them what to do. This view of mathematics—and what is involved in learning it—is inconsistent with mathematics as a discipline and with the learning theories described previously. Therefore, it is a worthwhile goal to transform your classroom into a mathematical community of learners who interact with each other and with the teacher as they share ideas and results, compare and evaluate strategies, challenge results, determine the validity of answers, and negotiate ideas. The rich interaction in such a classroom increases opportunities for productive engagement and reflective thinking about relevant mathematical ideas, resulting in students developing a relational understanding of mathematics.

A second difference between the two classrooms is the learning goals. Both teachers might write "understand fraction division" as the objective for the day. However, what is captured in the word "understand" is very different in each setting. In the first classroom, the teacher's goal is for students to connect fractions and division to what they already know. In the second classroom, understanding is connected to being able to carry out the standard algorithm. The learning goals, and more specifically how the teacher interprets the meaning behind the learning goals, affect what students learn.

These lessons also differ in terms of how accessible they are—and this, in turn, affects who learns the mathematics. The first lesson is differentiated in that it meets students where they are in their current understanding. When a task is presented as "solve this in your own way," it has multiple entry points, meaning it can be approached in a variety of ways. Consequently, students with different prior knowledge or learning strategies can figure out a way to solve the problem. This makes the task accessible to more learners. Then, as students observe strategies that are more efficient than their own, they develop new and better ways to solve the problem.

In the second classroom, everyone has to do the problem in the same way. Students do not have the opportunity to apply their own ideas or to see that there are numerous ways to solve the problem. This may deprive students who need to continue working on the development of basic ideas of fractions or division, as well as students who could easily find one or more ways to do the problem if only they were asked to do so. The students in the second classroom are also likely to use the same method to divide all fractions instead of looking for more efficient ways to divide based on the meanings of division and relationships between numbers. For example, they are likely to divide $\frac{2}{3}$ by 2 using the standard algorithm instead of thinking that $\frac{2}{3} \div 2$ means that you divide $\frac{2}{3}$ into two equal parts, each of which is $\frac{1}{3}$. Recall in the discussion of learning theory the importance of building on prior knowledge and learning from others. In the first classroom, student-generated strategies, multiple approaches, and discussion about the problem represent the kinds of strategies that enhance learning for a range of learners.

Students in both classrooms will eventually succeed at dividing fractions, but what they learn about fractions and division—and about doing mathematics—is quite different. Understanding and doing mathematics involves generating strategies for solving problems, applying those approaches, seeing if they lead to solutions, and checking to see whether answers make sense. These activities were all present in the first classroom, but not in the second. Consequently, students in the first classroom, in addition to successfully dividing fractions, will develop richer mathematical understanding, become more flexible thinkers and better problem solvers, remain more engaged in learning, and develop more positive attitudes toward learning mathematics.

Mathematics Classrooms That Promote Understanding

Three of the most common types of teaching are direct instruction, facilitative methods (also called a *constructivist approach*), and coaching (Wiggins & McTighe, 2005). With direct instruction, the teacher usually demonstrates or models, lectures, and asks questions that are convergent or closed-ended in nature. With facilitative methods, the teacher might use investigations and inquiry, cooperative learning, discussion, and questions that are more open-ended. In coaching, the teacher provides students with guided practice and feedback that highlights ways to improve their performances.

You might be wondering which type of teaching is most appropriate if the goal is to teach mathematics for understanding. Unfortunately, there is no definitive answer because there are times when it is appropriate to engage in each of these types of teaching. Your approach depends on your instructional goals, the learners, and the situation. Some people believe that all direct instruction is ineffective because it ignores the learner's ideas and removes the productive struggle or opportunity to learn. This is not necessarily true. A teacher who is striving to teach for understanding can share information by using direct instruction as long as that information does not remove the need for students to reflect on and productively struggle with the situation at hand. In other words, regardless of instructional design, the teacher should not be doing the thinking, reasoning, and connection building—it must be the students who are engaged in these activities.

Regarding facilitative or constructivist methods, remember that constructivism is a theory of learning, not a theory of teaching. Constructivism helps explain how students learn—by developing and modifying ideas (schemas) and by making connections between these ideas. Students can learn as a result of different kinds of instruction. The instructional approach chosen should depend on the ideas and relationships students have already constructed. Sometimes students readily make connections by listening to a lecture (direct instruction). Sometimes they need time to investigate a situation so they can become aware of the different ideas at play and how those ideas relate to one another (facilitative). Sometimes they need to practice a skill and receive feedback on their performance to become more accurate (coaching). No matter which type of teaching is used, constructivism and sociocultural theories remind us as teachers to continually wonder whether our students have truly developed the given concept or skill, connecting it to what they already know. By shedding light on what and how our students understand, assessment can help us determine which teaching approach may be the most appropriate at a given time.

The essence of developing relational understanding is to keep the students' ideas at the forefront of classroom activities by emphasizing the process standards, mathematical proficiencies, and the Standards for Mathematical Practice. This requires that the teacher create a classroom culture in which students can learn from one another. Consider the following

features of a mathematics classroom that promote understanding (Chapin, O'Conner, & Anderson, 2009; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997; Hoffman, Breyfogle, & Dressler, 2007). In particular, notice who is doing the thinking, the talking, and the mathematics—the students.

- *Students' ideas are key.* Mathematical ideas expressed by students are important and have the potential to contribute to everyone's learning. Learning mathematics is about coming to understand the ideas of the mathematical community.
- *Opportunities for students to talk about mathematics are common*. Learning is enhanced when students are engaged with others who are working on the same ideas. Encouraging student-to-student dialogue can help students think of themselves as capable of making sense of mathematics. Students are also more likely to question each other's ideas than the teacher's ideas.
- *Multiple approaches are encouraged.* Students must recognize that there is often a variety of methods that will lead to a solution. Respect for the ideas shared by others is critical if real discussion is to take place.
- *Mistakes are good opportunities for learning.* Students must come to realize that errors provide opportunities for growth as they are uncovered and explained. Trust must be established with an understanding that it is all right to make mistakes. Without this trust, many ideas will never be shared.
- *Math makes sense.* Students must come to understand that mathematics makes sense. Teachers should resist always evaluating students' answers. In fact, when teachers routinely respond, "Yes, that's correct" or "No, that's wrong," students will stop trying to make sense of ideas in the classroom and discussion, and learning will be curtailed.

Teaching Tip

Listen carefully to students as they talk about what they are thinking and doing as they engage in a mathematical task. If they respond in an unexpected way, try to avoid imposing *your* ideas onto their ideas. Ask clarifying questions to try to make sense of the sense your students are making!

To create a climate that encourages mathematics understanding, teachers must first provide explicit instruction on the ground rules for classroom discussions. Second, teachers may need to model the type of questioning and interaction that they expect from their students. Direct instruction would be appropriate in such a situation. The crucial point in teaching for understanding is to highlight and use students' ideas to promote mathematical proficiency.

Most people go into teaching because they want to help students learn. It is hard to think of allowing—much less planning for—the students in your classroom to struggle. Not to show them a solution when they are experiencing difficulty seems almost counterintuitive. If our goal is relational understanding, however, the struggle is part of the learning and teaching becomes less about the teacher and more about what the students are doing and thinking.

Keep in mind that you too are a learner. Some ideas in this book may make more sense to you than others. Others may even create dissonance for you. Embrace this feeling of disequilibrium and unease as an opportunity to learn—to revise your perspectives on mathematics and on the teaching and learning of mathematics as you deepen your understanding so that you can, in turn, help your students deepen theirs.

Stop and Reflect

Look back at the chapter and identify any ideas that make you uncomfortable or that challenge your current thinking about mathematics or about teaching and learning mathematics. Try to determine why these ideas challenge you or raise questions for you. Write these ideas down and revisit them later as you read and reflect further. ■